SPRING 2024 MATH 590: EXAM I SOLUTIONS

Name:

Throughout V will denote a vector space over $F = \mathbb{R}$ or $\mathbb{C}$.

(I) True-False. Write true or false next to each of the statements below. (3 points each)

(a) $\mathbb{R}^{17}$ can be spanned by 19 vectors. True. One can always add redundant vectors to any spanning set.

(b) If V is a finite dimensional vector space, then V has only finitely many subspaces. False. There are infinitely many distinct lines through the origin in $\mathbb{R}^2$.

(c) Ten linearly independent vectors in $\mathbb{R}^{10}$ form a basis for $\mathbb{R}^{10}$. True. Discussed many times in class.

(d) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation. If $\ker(T) = 0$, then $\mathrm{im}(T) = \mathbb{R}^4$. False. By the Rank plus Nullity theorem, $\mathrm{im}(T)$ has dimension three, so it cannot equal $\mathbb{R}^4$.

(e) Suppose $V = \mathrm{Span}\{v_1, v_2, v_3, v_4\}$ and $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = \vec{0}$, with each $a_i \in F$ and $a_1 \neq 0$. Then $V = \mathrm{Span}\{v_2, v_3, v_4\}$. True, since we can solve for v_1 in terms of v_2, v_3, v_3.

(II) State the indicated definition, proposition or theorem. (5 points each)

(a) State the Rank plus Nullity Theorem and be sure **define all terms used in your statement**. (10 points)

Solution. Let $T : V \to W$ be a linear transformation, with V finite dimensional over F. Then

$$\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T)).$$

$\dim(V)$ is the number of elements in any basis for V.

$\ker(T) = \text{kernel of } T = \{v \in V \mid T(v) = 0\}$.

$\mathrm{im}(T) = $ the image of T$= \{w \in W \mid w = T(v), \text{for soem } v \in V\}$.

(b) Let $T : V \to W$ be a linear transformation, $\alpha = \{v_1, \ldots, v_n\}$ a basis for V and $\beta = \{w_1, \ldots, w_m\}$ a basis for W. Define (and not give a formula for) $[T(v)]_\beta$. (5 points)

Solution. If $T(v) = b_1w_1 + \cdots + b_mw_m$, then $[T(v)]_\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in F^m$.

(III) Short Answer. (15 points each)

(a) Suppose $v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ -4 \\ 0 \\ 5 \end{pmatrix}$. Write the matrix equation you would solve to determine if these vectors are linearly independent and **explain what a possible solution to this equation means**. Do not work out the details of solving the matrix equation.

Solution. One considers the matrix equation

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -4 \\ 0 & 1 & 0 \\ -2 & 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If the only solution is $x = y = z = w = 0$, then the given vectors are linearly independent. On the other hand, if $x = a, y = b, z = c$ is a non-zero solution, then $av_1 + bv_2 + cv_3 = \vec{0}$ is a dependence relation on v_1, v_2, v_3.

(b) Let $V = \mathrm{M}_{2\times 2}(\mathbb{R})$ and $T : V \to \mathbb{R}$ be the linear transformation $T(A) = \mathrm{trace}(A)$. Verify the Rank plus Nullity theorem.

Solution. We compute the kernel and image of T. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the kernel of T. Then $a + d = 0$, so $d = -a$. Thus, $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Note that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ belong to the kernel of T, so they span the kernel of T. Moreover, given a linear combination $r \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + s \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + t \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, we have $\begin{pmatrix} r & s \\ t & -r \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $r = s = t = 0$, showing that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are linearly independent, and thus form a basis for the kernel of T. Therefore, $\dim(\ker(T)) = 3$. Note that for any $r \in \mathbb{R}$, $T \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = r$, so that T is an onto transformation. Thus, $\mathrm{im}(T) = \mathbb{R}$, so that $\dim(\mathrm{im}(T)) = 1$. Therefore, we have

$$4 = \dim(V) = 3 + 1 = \dim(\ker(T)) + \dim(\mathrm{im}(T)),$$

confirming the Rank plus Nullity theorem.

Comment. Finding the basis for the kernel in this problem is very similar to Example 4 in Lecture 7, where we find a basis for all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying $3a + 2d = 0$.

(c) Suppose $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(v) = Av$, for all $v \in \mathbb{R}^2$. Let α denote the standard basis for $\mathbb{R}^2$ and set $\beta := \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. Explain why β is a basis for $\mathbb{R}^2$, then state the change of basis formula as it applies here, and use it to calculate $[T]_\beta^\beta$. Note: Do not calculate $[T]_\beta^\beta$ directly.

Solution. First note that β is a basis for $\mathbb{R}^2$, since $\det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 1 \neq 0$. The change of basis formula states that $[T]_\beta^\beta = [I]_\alpha^\beta \cdot [T]_\alpha^\alpha \cdot [I]_\beta^\alpha$.

As seen many times in class, we have $[T]_\alpha^\alpha = A$. Moreover, $[I]_\beta^\alpha = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, since α is the standard basis for $\mathbb{R}^2$. Since $[I]_\alpha^\beta = ([I]_\beta^\alpha)^{-1}$, we have $[I]_\alpha^\beta = \frac{1}{1} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. Thus,

$$[T]_\beta^\beta = [I]_\alpha^\beta \cdot [T]_\alpha^\alpha \cdot [I]_\beta^\alpha = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Comment. This almost the same type of problem as Practice Problem 3, except, here, you are not required to calculate $[T]_\beta^\beta$ in two different ways.

(IV) Proof problem. (25 points) For the linear transformations $T : V \to W$ and $S : W \to U$, and bases $\alpha \subseteq V, \beta \subseteq W, \gamma \subseteq U$, **state and prove** the formula relating the matrices of S and T to the matrix of ST with respect to the given bases.

Solution. We are required to prove that $[ST]_\alpha^\gamma = [S]_\beta^\gamma \cdot [T]_\alpha^\beta$.

We take $\alpha := \{v_1, \dots, v_n\}$ and $\beta := \{w_1, \dots, w_m\}$, and set $[T]_\alpha^\beta := A = (a_{ij})$ and $[S]_\beta^\gamma := B = (b_{ij})$. Let $B_1, \dots, B_m$ denote the columns of B.

On the one hand, the jth column of $[ST]_\alpha^\gamma$ is $[ST(v_j)]_\gamma$. On the other hand,

$$\begin{aligned}
[ST(v_j)]_\gamma &= [S(a_{1j}w_1 + \cdots + a_{mj}w_m)]_\gamma \\
&= [a_{1j}S(w_1) + \cdots + a_{mj}S(w_m)]_\gamma \\
&= a_{1j}[S(w_{1j})]_\gamma + \cdots + a_{mj}[S(w_m)]_\gamma \\
&= a_{1j}B_1 + \cdots + a_{mj}B_m \\
&= B \cdot \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix},
\end{aligned}$$

which is the jth column of $B \cdot A$, i.e., the jth column of $[S]_\beta^\gamma \cdot [T]_\alpha^\beta$. Thus the matrices $[ST]_\alpha^\gamma$ and $[S]_\beta^\gamma \cdot [T]_\alpha^\beta$ have the same columns and are therefore equal.